SAA Working Group on Yield Curves

First Results

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EPFL and Swiss Finance Institute

SAA Annual Meeting
Andermatt, 26 August 2022

SAA Working Group on Yield Curves

Goals:

- Provide technical input to FINMA-AG Zinskurven
- Application of **Kernel Ridge (KR) method** developed in Filipović–Pelger–Ye (2022) "Stripping the discount curve – a robust machine learning approach" to CHF, EUR, USD, GBP, JPY
- Explore data sources according to criteria availability, quality, completeness, cost
- Further development of KR method towards multi-currency learning

Organisation:

- Lead by Lutz Wilhelmy and Damir Filipović
- WG members from CH industry (Baloise, Generali, Mobiliar, Swiss Life, Swiss Re, Zurich) and academia (EPFL)
- Kick-off in June 2022

Principles of Yield Curve Estimation

- Simple and fast to implement
- Transparent and reproducible
- Data-driven
- Precise representation of the term structure, taking into account all market signals
- Robust to outliers and data selection choices
- Flexible for integration of external views: exogenous points, choice of weights
- Consistent with finance principles

Methods in scope

- Nelson–Siegel–Svensson (NSS) (1987, 1994): parsimonious parametric, highly non-convex optimization
- SNB Nelson–Siegel–Svensson (2002): NSS with parameter constraints to match overnight rate (since 2021 SARON 1M-swap)
- Smith–Wilson (2001): interpolation-extrapolation method, Solvency II standard, used for SST since 2012 based on SNB NSS
- KR method (2022): robust kernel ridge regression. Paper is available at SSRN: `https://ssrn.com/abstract=4058150`

Outline

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Ingredients

- **Unobserved discount curve** $g(x)$ = fundamental value of a non-defaultable zero-coupon bond with time to maturity x
- **Observed**: M fixed income securities with
  - cash flow dates $0 < x_1 < \cdots < x_N$
  - $M \times N$ cash flow matrix C
  - noisy ex-coupon prices $P = (P_1, \ldots, P_M)^\top$
- No-arbitrage pricing relation:

$$P_i = \underbrace{C_i g(\boldsymbol{x})}_{\text{fundamental value}} + \underbrace{\epsilon_i}_{\text{pricing error}},$$

  where $\boldsymbol{x} = (x_1, \ldots, x_N)^\top$ and $g(\boldsymbol{x}) = (g(x_1), \ldots, g(x_N))^\top$

- ϵ_i: deviations from fundamental value, due to market imperfections (no deep, liquid, transparent market) and data errors

Estimation problem

Problem: Minimize pricing errors for some exogenous weights ω_i:

$$\min_g \left\{ \sum_{i=1}^{M} \omega_i \left(P_i - C_i g(\boldsymbol{x})\right)^2 \right\}$$

- Observe only $M \approx 25$ bonds, need to estimate $N \approx 15{,}000$ (40 years $\times$ 365 days) discount bond prices
- Any estimation approach imposes regularizing assumptions to limit the number of parameters
- Existing approaches ad-hoc assumptions $\Rightarrow$ misspecified form

KR approach: Smoothness regularization

- Limits to arbitrage require a sufficiently smooth curve, as large sudden changes imply risk-free extreme payoffs

Smooth discount curves

General measure of smoothness for functions

$$\|g\|_{\alpha,\delta} = \left(\int_0^\infty \left(\delta g'(x)^2 + (1-\delta) g''(x)^2 \right) \mathrm{e}^{\alpha x} \, dx \right)^{\frac{1}{2}}$$

- Curvature $g''(x)^2$: penalizing avoids kinks
- Tension $g'(x)^2$: penalizing avoids oscillations
- Maturity weight $\alpha \geq 0 \Rightarrow$ corresponds to infinite-maturity yield
- Tension parameter $\delta \in [0,1)$ balances tension and curvature

$\Rightarrow$ Work with extremely large hypothesis space of discount curves given by the set $\mathcal{G}_{\alpha,\delta}$ of twice differentiable functions $g : [0,\infty) \to \mathbb{R}$ with $g(0) = 1$ and finite smoothness measure $\|g\|_{\alpha,\delta} < \infty$

Fundamental estimation problem

Fundamental optimization problem:

$$\min_{g \in \mathcal{G}_{\alpha,\delta}} \left\{ \underbrace{\sum_{i=1}^{M} \omega_i (P_i - C_i g(\boldsymbol{x}))^2}_{\text{pricing error}} + \lambda \underbrace{\|g\|_{\alpha,\delta}^2}_{\text{smoothness}} \right\} \tag{1}$$

- Smoothness parameter $\lambda > 0$: Trade-off between pricing errors and smoothness
- Exogenous weights $0 < \omega_i \leq \infty$ ($\omega_i = \infty$ is exact pricing): we set ω_i to duration weights $\Rightarrow$ approximate yield fitting
- Problem completely determined up to the three parameters α, δ, λ selected empirically via cross-validation to minimize pricing errors out-of-sample $\Rightarrow$ fully data-driven.

Kernel Ridge (KR) solution

The KR solution to fundamental problem (1) is given by:

$$\hat{g}(x) = 1 + \sum_{j=1}^{N} k(x, x_j)\beta_j, \quad \text{where} \quad \beta = C^\top (C\boldsymbol{K}C^\top + \Lambda)^{-1}(P - C\mathbf{1}),$$

for $N \times N$-kernel matrix $\boldsymbol{K}_{ij} = k(x_i, x_j)$, and $\Lambda = \mathrm{diag}(\lambda/\omega_1, \dots, \lambda/\omega_M)$

- Simple closed-form solution, easy to implement
- Basis functions $k(., x_j)$ are determined by smoothness measure
- Discount bonds are portfolios of coupon bonds $\Rightarrow$ Immunization
- Nelson–Siegel–Svensson and Smith–Wilson discount curves are special cases of KR framework for specific parameter choices.

Special curves: Nelson–Siegel–Svensson

Nelson–Siegel–Svensson (NSS) assume a parametric forward curve

$$f_{NSS}(x) = \gamma_0 + \gamma_1 \mathrm{e}^{-\frac{x}{\tau_1}} + \gamma_2 \frac{x}{\tau_1} \mathrm{e}^{-\frac{x}{\tau_1}} + \gamma_3 \frac{x}{\tau_2} \mathrm{e}^{-\frac{x}{\tau_2}}$$

for real parameters $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ and $\tau_1, \tau_2 > 0$.

Lemma 1.1.

The NSS curve $g_{NSS}(x) = \mathrm{e}^{-\int_0^x f_{NSS}(t)\,dt}$ *lies in* $\mathcal{G}_{\alpha,\delta}$*, if* $\alpha < 2\gamma_0$.

Special curves: Smith–Wilson

Smith–Wilson assume discount curves of the form

$$g_{SW}(x) = \mathrm{e}^{-y_\infty x} g_0(x), \quad y_\infty := \log(1 + UFR),$$

for some $g_0 \in \mathcal{G}_{0,1/2}$ and ultimate forward rate $UFR > 0$.

- Assume exact pricing up to last liquid point (minimal regularity)
- Insurance industry standard in Europe
- Used in the regulatory Solvency II framework

Lemma 1.2.

The Smith–Wilson curve g_{SW} lies in $\mathcal{G}_{\alpha,\delta}$, if $\alpha < 2y_\infty$.

Bayesian perspective and distribution theory

Assume g is a Gaussian process with prior distribution

$$g(\boldsymbol{x}) \sim \mathcal{N}\left(m(\boldsymbol{x}), k(\boldsymbol{x}, \boldsymbol{x}^\top)\right),$$

with pricing errors $\boldsymbol{\epsilon} \sim \mathcal{N}(0, \Sigma^\epsilon)$ for $\Sigma^\epsilon = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_M^2)$.

Theorem 1.3 (Bayesian perspective).

If the prior mean function $m(x) = 1$ and pricing error variance $\sigma_i^2 = \lambda/\omega_i$, then

1. *the posterior mean function equals the KR estimated discount curve,*
2. *the posterior distribution is Gaussian with known posterior variance.*

$\Rightarrow$ We obtain a confidence range for the discount curve and securities

Outline

Data, estimation and evaluation

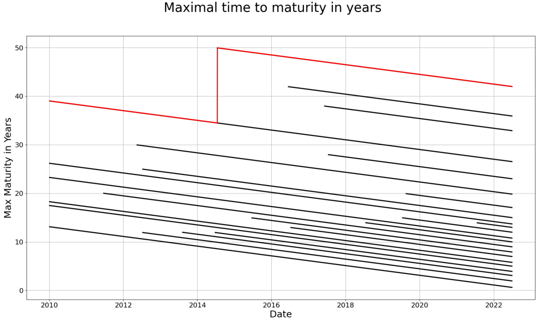
CH Confederation bonds:

- CH Confederation bond data from SNB public source
- Daily ex-dividend (clean) mid-prices (adjusted for AI)
- Sampling period: January 2010 to June 2022 (150 months)
- Total of 22 issues of Confederation bonds

Estimation and evaluation:

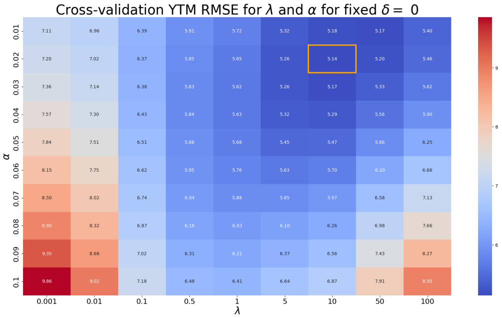
- Estimation without bonds maturing in less than 3M
- In-sample evaluation with all bonds
- Cross-sectional out-of-sample with LOO cross-validation
- Root-mean-squared errors (RMSE) for yields and relative prices

Data: maturity ranges



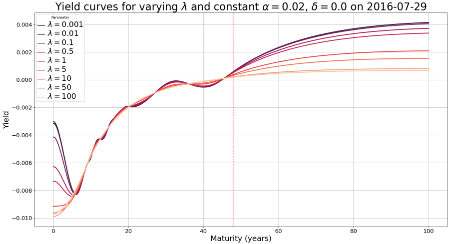
- Figure shows the time to maturity of the data. Red: maximum
- Unequal maturity distribution: long maturities underrepresented
- Unbalanced panel: > 40 years only available after July 2014

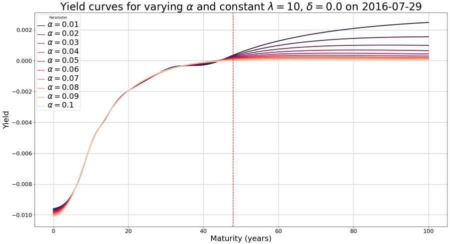
Cross-validation for hyper-parameters α, δ, λ



- Figure shows average cross-validation YTM fitting error (in bps)
- Optimal values (baseline choice): $\lambda = 10$, $\alpha = 0.02$, $\delta = 0$
- Results are robust to the choice of hyper-parameters

Illustration: yield curve estimates as fct of parameters





- Representative example day: 2016-07-29
- Effect of λ: less curvature $\Rightarrow$ bias-variance tradeoff
- Effect of α: only affects long maturities $\Rightarrow$ $\alpha =$ infinite-maturity yield

$\Rightarrow$ Extrapolation is a choice and not verifiable on observed data

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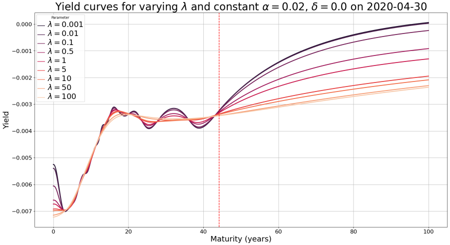

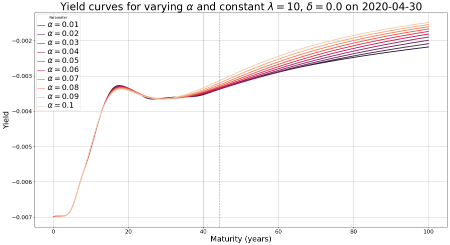

- Representative example day: 2020-04-30
- Effect of λ: less curvature $\Rightarrow$ bias-variance tradeoff
- Effect of α: only affects long maturities $\Rightarrow$ $\alpha =$ infinite-maturity yield

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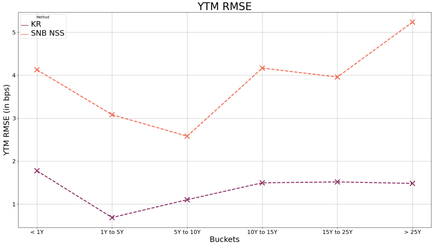
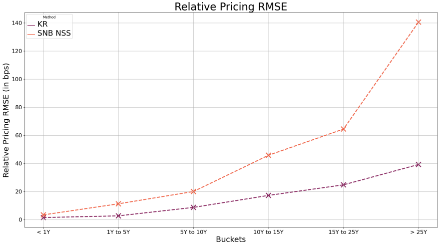
Benchmark models

- Nelson–Siegel–Svensson (NSS): non-convex optimization $\Rightarrow$ not (easily) reproducible
- SNB Nelson–Siegel–Svensson: NSS with parameter constraints
- SST curves (since 2021): Smith–Wilson method based on SNB NSS

Average in-sample pricing errors for different maturities



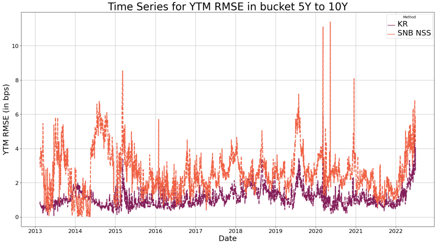
YTM RMSE

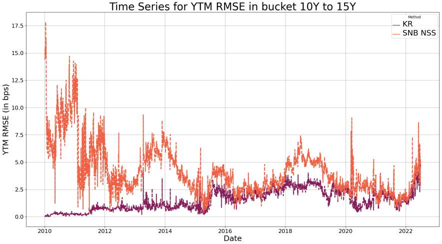


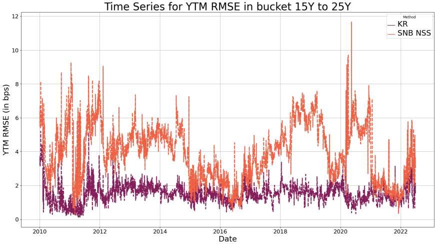
Relative Pricing RMSE

- In-sample evaluation with all bonds
- KR dominates all benchmark methods along all maturities
- KR has smallest yield and pricing errors for all bonds, also over time . . .

Time series for in-sample YTM RMSE per bucket







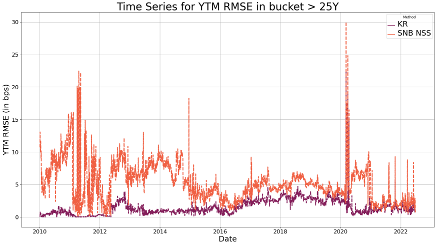
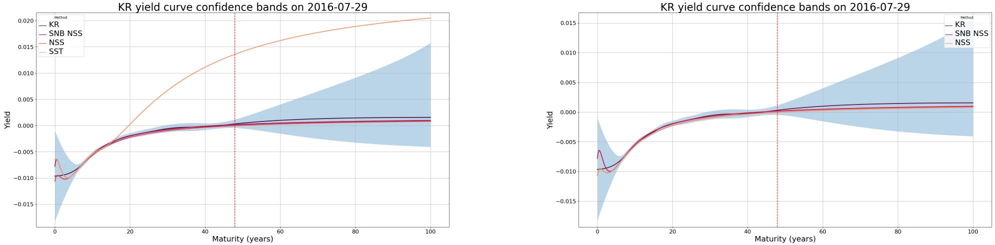


Illustration: yield curve estimates of different methods



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- NSS curves not flexible and excessive curvature in the short end
- SST curve biased by UFR (left panel)
- 99% confidence intervals wider for maturities with more dispersed or less observed prices

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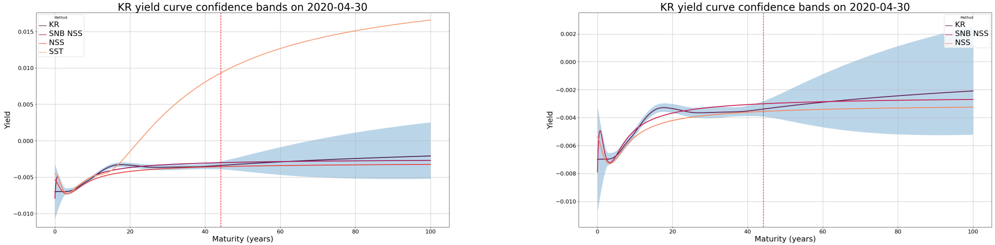

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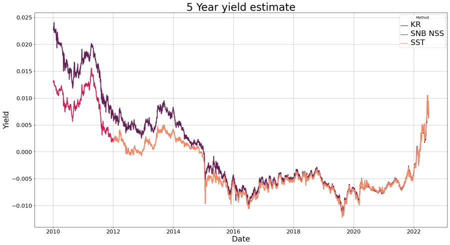
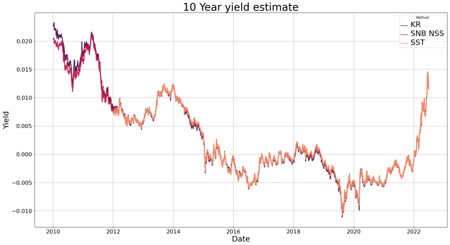
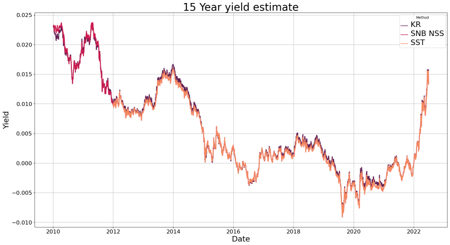
Short and long maturity yield estimates over time

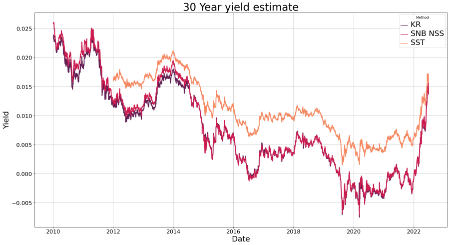




5Y and 10Y yield estimates: similar volatility

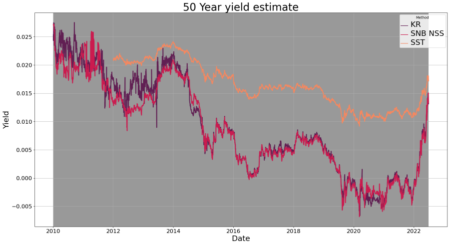
Short and long maturity yield estimates over time

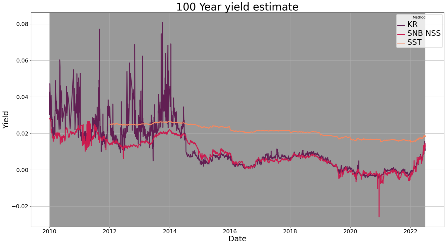




15Y and 30Y yield estimates: similar volatility

Short and long maturity yield estimates over time





50Y and 100Y yield estimates:
extrapolations can be very volatile ⇒ exogenous points necessary

Conclusion and outlook

- KR method satisfies all principles of yield curve estimation
- KR method dominates NSS-SNB and SST curves: easily reproducible and most precise representation of the term structure
- Extrapolation to 50Y and beyond: requires exogenous input
- E.g., multi-curve learning CHF, EUR, USD, GBP, JPY, learn about CHF curve from long maturities of other currencies (e.g., Austria 100Y Government Bond) $\Rightarrow$ ongoing research

Backup: list of WG members

- Lutz Wilhelmy (Swiss Re)
- Damir Filipović (EPFL)
- Nicolas Camenzind (EPFL)
- Andreas Lutz (Baloise)
- Dominik Stich (Baloise)
- Philipp Keller (Generali)
- Oliver Strub (Mobiliar)
- Urs Müller (Swiss Life)
- Tsunehiro Tsujimoto (Swiss Re)
- Jozef Minar (Zurich)